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Large Numbers Hypothesis. III. Kinetic Theory, Statistical Physics, and Thermodynamics

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Dirac's large numbers hypothesis (LNH) is incorporated into kinetic theory, statistical physics, and thermodynamics using the self-consistent formalism of units covariance. The ingeodesic equation and matter creation introduce modifications of the most fundamental laws of the subject. Liouville's theorem no longer holds, the Boltzmann equation is modified, as is the H-theorem. This affects the second law of thermodynamics in that for canonical LNH neither reversible nor adiabatic processes are possible (as expected). A significant result is that the collision terms have the same form as in standard physics. This means that equilibrium distribution functions are identical to those of standard physics, as required for self-consistency with the precepts of LNH. The net effect of LNH is as though all matter in our Universe were weakly coupled to a large heat bath.

1. INTRODUCTION

This paper is the third in a series seeking to explore the consequences for physics of developing a viable, self-consistent, physical theory incorporating Dirac's (1937) large numbers hypothesis (LNH). In Papers I and II (Adams, 1982, 1983) LNH was presented, the guiding principle of units covariance was developed, a scalar "field" $\varphi(x)$ mediating all LNH phenomena was introduced, and the theory of electromagnetic radiation was explored. In this paper I develop kinetic theory, statistical physics, and thermodynamics incorporating LNH in the units covariant formalism (Adams, 1982).

Anticipated applications of this LNH formalism include stellar evolution and the evolution of the cosmic "soup" in isotropic, homogeneous cosmological models. While such applications do not involve kinetic theory *per se*, in order to sensibly formulate statistical physics and thermodynamics in a self-consistent way I start with kinetic theory. This is the natural approach since the single-particle equations of motion for both matter particles and photons are known from Papers I and II.

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The modifications of kinetic theory are brought about by the ingeodesic equation and by matter creation. One finds that the phase space volume element is no longer preserved (Liouville's theorem fails). Consequently Boltzmann's equation is modified. A significant result is that the collision terms have the same form as in standard physics. This means that equilibrium distribution functions are identical to those of standard physics. As required for self-consistency with the precepts of LNH, the effect of φ is to change the dynamics of statistical ensembles, not their kinematics. Finally, the H-theorem is modified, which significantly affects the second law of thermodynamics. The net effect of LNH is as though all matter in our Universe were weakly coupled to a large heat bath.

Having incorporated LNH into kinetic theory in the units covariant formalism one can obtain statistical physics and thermodynamics in the standard manner (Stewart, 1971). As expected, matter creation and the in-geodesic equation combine to indicate that for canonical LNH (Paper II) neither reversible processes nor adiabatic processes exist. Those processes which minimize the entropy change are referred to as ideal.

For convenience and ease of presentation the work here is applicable only to strictly atomic systems, i.e., those systems which have constant mass and replicate in A units. Hence classical systems, i.e., those systems having constant mass in G units but which do not replicate, such as star clusters, will satisfy similar but not identical equations. The difference arises in the normalizing parameters entering the analysis. For atomic systems these are Planck's constant and Boltzmann's constant. The difference amounts to deciding in what units (G units or A units) these parameters are constant. For classical systems they must be constant in G units while for atomic systems they must be constant in A units.

Finally, several simple applications are presented to illustrate use of the formalism and to verify self-consistency with previously established results. The reader unfamiliar with sign conventions, notation, or the units covariant formalism is referred to Paper I for details.

2. KINETIC THEORY[3]

2.1. Phase Space. The world line of a single particle is determined by equations of motion in the form

$$\frac{dx^\alpha}{d\lambda} = p^\alpha \tag{1a}$$

$$\frac{dp^\alpha}{d\lambda} = F^\alpha(x, p) \tag{1b}$$

[3]Stewart (1971); Ehlers (1971).

where, for example, for a free particle satisfying

$$p^{\mu}{}_{*\alpha} p^{\alpha} = - p^{\mu} p^{\alpha} (\ln N)_{,\alpha} \tag{2}$$

$$F^{\alpha}(x, p) = - p^{\mu} p^{\lambda} \Gamma^{\alpha}_{\mu\lambda} - p^{\alpha} p^{\mu} (\ln N)_{,\mu} - (2-g) p^{\alpha} p^{\mu} (\ln \beta)_{,\mu}$$
$$+ p^{\mu} p^{\lambda} g_{\mu\lambda} g^{\alpha\rho} (\ln \beta)_{,\rho} \tag{3}$$

with $\dot{N}$ the particle replication rate. Notice that (2) and (3) are equally valid for photons. The path parameter λ is uniquely determined up to an additive constant by the requirement that p^{α} be the four-momentum (Adams, 1983), regardless of whether $p^{\alpha} p_{\alpha} > 0$ or $p^{\alpha} p_{\alpha} = 0$. If $p^{\alpha} p_{\alpha} > 0$ then $d\tau = (p^{\alpha} p_{\alpha})^{1/2} \, d\lambda$ is the particle proper time.

The equations of motion (1) define a vector field

$$L \equiv p^{\alpha} \frac{\partial}{\partial x^{\alpha}} + F^{\alpha}(x, p) \frac{\partial}{\partial p^{\alpha}} \tag{4}$$

on the eight-dimensional phase space R_p of the system. L is called the Liouville operator and since $L \sim d/d\lambda$, $L(f)$ determines the variation of f along a test particle path for any f. The phase flow generated by L is the set of integral curves of (1).

Define

$$\hat{h}(x, p) \equiv g_{\mu\alpha}(x) p^{\mu} p^{\alpha} \tag{5}$$

Use of (4) and (3) shows that

$$L(\hat{h}) = \hat{h}[(g-1) \ln \beta - \ln N]_{,\alpha} p^{\alpha} \tag{6}$$

or $L(m_0) = 0$, where

$$\hat{h} = m_0 (\beta^{g-1}/N) \tag{7}$$

For quantum matter particles the number of particles varies as (Adams, 1982)

$$N = N_{m_0} (\varphi/\varphi_0)^{g-1} \tag{8}$$

so appropriate normalization turns (7) into the familiar (Adams, 1982)

$$m = m_A (\beta/\varphi)^{g-1} \tag{9}$$

A seven-dimensional hypersurface R_m of R_p is defined by $m_0 = \text{const}$ and is generated by all those orbits in phase space belonging to m_0. R_m is the phase space for particles of mass m_0. Since $L(m_0) = 0$ the vector L is tangent to R_m.

From now on I deal with particles of a given mass m_0 only. Hence one deals with the seven-dimensional R_m as the phase space of the system. A coordinate-independent volume element on R_m is

$$\Omega \equiv \eta \wedge \pi \tag{9a}$$

$$\eta \equiv \frac{1}{4!}(-g)^{1/2}\varepsilon_{\mu\alpha\sigma\lambda}\, dx^{\mu}\, dx^{\alpha}\, dx^{\sigma}\, dx^{\lambda} \to d^4x \tag{9b}$$

$$\pi \equiv \frac{2}{4!}H(p^0)\delta(p^{\lambda}p_{\lambda} - m^2)(-g)^{1/2}\varepsilon_{\mu\alpha\sigma\lambda}\, dp^{\mu}\, dp^{\alpha}\, dp^{\sigma}\, dp^{\lambda} \to \frac{d^3p}{E} \tag{9c}$$

where $\varepsilon_{\mu\alpha\sigma\lambda}$ is the alternating symbol and the arrows denote the form taken in a locally orthonormal coordinate system.

The final volume element needed is the volume element for six-dimensional hypersurfaces in R_m. Since L is tangent to R_m this is obtained by contracting Ω with L to get

$$\omega \equiv L \cdot \Omega = p^{\alpha}\sigma_{\alpha} \wedge \pi \tag{10}$$

where σ_{α} is a three-dimensional hypersurface element defined by

$$\sigma_{\alpha} \equiv \frac{1}{3!}(-g)^{1/2}\varepsilon_{\alpha\mu\sigma\lambda}\, dx^{\mu}\, dx^{\sigma}\, dx^{\lambda} \tag{11}$$

If the hypersurface of (10) is contained in the local, instantaneous rest space of an observer at x^{μ} with four-velocity u^{α}, and if the momentum hypersurface of (10) is small and contains p^{α}, then from (9c)

$$\omega = p^{\alpha}u_{\alpha}dV d^3p/E = d^3x\, d^3p \tag{12}$$

which is the ordinary differential phase space volume.

In standard physics $d\omega = 0$ (Liouville's theorem) but this is no longer true in this theory containing LNH. In fact from Paper II one has

$$\begin{aligned} d\omega &= -3d\ln N \wedge \omega + (3g-6)d\ln\beta \wedge \omega \\ &= -3[\ln N + (2-g)\ln\beta]_{,\alpha}p^{\alpha}\Omega \end{aligned} \tag{13}$$

so that ω is not invariant with respect to the phase flow except for classical particles ($N = \text{const}$) in G units ($\beta = 1$). This point is crucial for what follows, especially since one expects to apply (13) in A units ($\beta = \varphi$).

2.2. Boltzmann Equation. Let ∂A be the six-dimensional boundary of the seven-dimensional region $A \subset R_m$. Let $H(x, p)$ be an arbitrary function on R_m. Then

$$\begin{aligned}\int_{\partial A} H\omega &= \int_A d(H\omega) = \int_A dH \wedge (L\cdot\Omega) + \int_A H\,d\omega \\ &= \int_A \{L(H) - 3[\ln N + (2-g)\ln\beta]_{,\alpha} p^\alpha H\}\Omega \qquad (14)\end{aligned}$$

where the first term of (14) follows as in standard physics and the last term comes from (13).

Now introduce the distribution function $f(x, p)$ on R_m. Then the average net number of "collisions" in A is equal to the average number of particles intersecting ∂A as

$$\bar{N}(\partial A) = \int_{\partial A} f\omega = \int_A \{L(f) - 3[\ln N + (2-g)\ln\beta]_{,\alpha} p^\alpha f\}\Omega \qquad (15)$$

by (14). If the phase space density of "collisions" is denoted $\mathcal{C}(f)$ then clearly

$$L(f) - 3[\ln N + (2-g)\ln\beta]_{,\alpha} p^\alpha f = \mathcal{C}(f) \qquad (16)$$

However, the term "collisions" includes not only physical collisions of particles which scatter them in to and out of A as in standard physics, but also net particle creation. This is clear from (15) since $\bar{N}(\partial A)$ is the net flux of particles through ∂A. Even if no physical collisions scatter particles in to or out of A, particle creation would mean that more particles leave A than enter A. Hence

$$\mathcal{C}(f) = (\ln N)_{,\alpha} p^\alpha f + C(f) \qquad (17)$$

where now $C(f)$ is the phase space density of particle collisions (no quotes, physical collisions). Therefore, one finds

$$L(f) = [4\ln N + (6-3g)\ln\beta]_{,\alpha} p^\alpha f + C(f) \qquad (18)$$

which is consistent with the collision-free result found in Paper II.

Equation (18) describes the evolution of the distribution function $f(x, p)$ provided one knows the functional $C(f)$ from an analysis of collision mechanisms. If the duration of each collision is very small compared to the time scales of β and N then the collisions can be assumed to occur instantaneously, i.e., the interparticle forces "causing" collisions can be approximated as hard core and the collisions are approximated as occurring at a point. In this case the collision functional $C(f)$ will have exactly the same form as in standard physics:

$$C_a(f_a) = \tfrac{1}{2} \sum_{b,c,d} \iiint (\hat{g}_a \hat{g}_b g_c g_d - g_a g_b \hat{g}_c \hat{g}_d) W_{ab \to cd} \pi_b \wedge \pi_c \wedge \pi_d$$

$$+ \sum_{b,c} \iint (\hat{g}_a \hat{g}_b g_c - g_a g_b \hat{g}_c) W_{ab \to c} \pi_b \wedge \pi_c$$

$$+ \tfrac{1}{2} \sum_{b,c} \iint (\hat{g}_a g_b g_c - g_a \hat{g}_b \hat{g}_c) W_{a \to bc} \pi_b \wedge \pi_c \tag{19a}$$

$$g_a \equiv h^3 f_a / r_a, \qquad \hat{g}_a \equiv 1 \pm g_a \tag{19b}$$

with f_a the distribution function of species a, h Planck's constant, r_a the spin degeneracy of particles of species a, the upper sign in (19b) is for Bose particles and the lower sign for Fermi particles, and W is a measure of the probability of occurrence of the particular collision. Notice that $W_{ab \to cd} = W_{cd \to ab}$ and that the sums in (19) extend over all species present. Equations (18) and (19) constitute the LNH generalization of Boltzmann's equation.

2.3. Moments of the Distribution Function, Entropy, and the *H*-Theorem. Various moments of the distribution function $f(x, p)$ can be defined as

$$n^\alpha \equiv \int p^\alpha f \pi \tag{20}$$

$$T^{\mu\alpha} \equiv \int p^\mu p^\alpha f \pi \tag{21}$$

etc. where n^α is the number density flux vector and $T^{\mu\alpha}$ is the energy tensor. If V is an arbitrary spacetime region with boundary ∂V which is contained in the seven-dimensional region A with six-dimensional boundary ∂A one has

$$\int_V \eta n^\alpha{}_{;\alpha} = \int_{\partial V} \sigma_\alpha n^\alpha = \int_{\partial V} \int p^\alpha f \sigma_\alpha \wedge \pi = \int_{\partial A} f\omega$$

$$= \int_A \{L(f) - 3[\ln N + (2-g)\ln\beta]_{,\alpha} p^\alpha f\}\Omega$$

$$= \int_V \eta \wedge \int [(\ln N)_{,\alpha} p^\alpha f + C(f)]\pi \tag{22}$$

from (20), (10), (14), and (18). Since V is arbitrary

$$n^\alpha{}_{;\alpha} = \int [(\ln N)_{,\alpha} p^\alpha f + C(f)]\pi$$

$$= (\ln N)_{,\alpha} n^\alpha + \int C(f)\pi \tag{23}$$

using (20) and the fact that N is independent of p^α (at least to lowest order) in the LNH formalism. From (20) and (21)

$$\Pi(n^\alpha) = -4, \qquad \Pi(T^{\mu\alpha}) = -g - 4 \tag{24}$$

using $\Pi(p^\alpha) = -g$ so

$$n^\alpha{}_{;\alpha} = n^\alpha{}_{*\alpha} = n^\alpha{}_{\|\alpha} = (\ln N)_{,\alpha} n^\alpha + \int C(f)\pi \tag{25}$$

Of course, (25) is just what one expects since one finds the standard physics result modified by the creation of new particles.

Let V, ∂V, A, and ∂A be as above and let $\xi_\mu(x)$ be an arbitrary smoothly differentiable covariant vector field. Then

$$\int_V \eta(\xi_\mu T^{\mu\alpha})_{;\alpha} = \int_A \{L(\xi_\mu p^\mu f) - 3[\ln N + (2-g)\ln\beta]_{,\alpha} p^\alpha \xi_\mu p^\mu f\}\Omega$$

$$= \int_V \eta \wedge \int \left[L(\xi_\mu p^\mu) f + (\ln N)_{,\alpha} p^\alpha \xi_\mu p^\mu f + C(f)\xi_\mu p^\mu\right]\pi \tag{26}$$

using linearity of the Liouville operator together with the same steps as led to (22). Since V is arbitrary

$$\xi_{\mu;\alpha} T^{\mu\alpha} + \xi_\mu T^{\mu\alpha}{}_{;\alpha} = \int L(\xi_\mu p^\mu) f\pi + (\ln N)_{,\alpha} \xi_\mu T^{\alpha\mu} + \xi_\mu \int C(f) p^\mu \pi \tag{27}$$

using (21). But from (4)

$$L(\xi_\mu p^\mu) = \xi_{\mu,\alpha} p^\alpha p^\mu + \xi_\alpha F^\alpha(x, p) \tag{28}$$

so since ξ_μ is arbitrary

$$T^{\mu\alpha}{}_{;\alpha} = (\ln N)_{,\alpha} T^{\alpha\mu} - \int p^\mu p^\alpha (\ln N)_{,\mu} f\pi - (2-g)\int p^\alpha p^\mu (\ln\beta)_{,\alpha} f\pi$$

$$+ g^{\mu\rho}(\ln\beta)_{,\rho}\int p^\lambda p_\lambda f\pi + \int C(f) p^\mu \pi$$

$$= -(2-g)(\ln\beta)_{,\rho}T^{\mu\rho} + g^{\mu\rho}(\ln\beta)_{,\rho}T^\lambda{}_\lambda + \int C(f)p^\mu\pi \tag{29}$$

using (3) and (21). But (29) can be written as

$$T^{\mu\alpha}{}_{*\alpha} = \int C(f) p^\mu \pi \tag{29b}$$

using (24). This is the units covariant generalization of the standard physics equation (semicolon-to-star rule). Notice that (29) does not contain φ explicitly since $T^{\mu\alpha}$ contains no explicit reference to particles. On the other hand, since n^α is explicitly particle density flux (25) does contain φ explicitly.

Finally, one introduces the entropy flux vector

$$s^\alpha \equiv -k\frac{r}{h^3}\int (g\ln g \mp \hat{g}\ln\hat{g})\, p^\alpha\pi \tag{30}$$

where g, $\hat{g}$, r, h and the sign convention are the same as in (19), and where k is Boltzmann's constant. Note that k has units of energy and hence power $1-g$ (see the discussion of k in Section 4 below). Quantum particles are expected to replicate freely only in the absence of quantum degeneracy effects, so take the dilute gas limit ($g \ll 1$) in (30) to get

$$s^\alpha = -k\frac{r}{h^3}\int (g\ln g - g)\, p^\alpha\pi$$

$$= k\int f p^\alpha\pi - k\int f(\ln g)\, p^\alpha\pi$$

$$= kn^\alpha - k\int f\ln(h^3 f/r)\, p^\alpha\pi \tag{31}$$

Now one is in a position to derive the units covariant version of the H-theorem. First define

$$a^\alpha \equiv \int f \ln(h^3 f/r)\, p^\alpha \pi \tag{32}$$

$$s^\alpha = k(n^\alpha - a^\alpha) \tag{33}$$

The proof follows that leading to (22):

$$\int_V \eta a^\alpha{}_{;\alpha} = \int_{\partial A} f \ln(h^3 f/r)\, \omega$$

$$= \int_A \{L[f \ln(h^3 f/r)] - 3[\ln N + (2-g)\ln\beta]_{,\alpha} p^\alpha f \ln(h^3 f/r)\}\Omega \tag{34}$$

so linearity of L, (18), (4), (9a), and

$$h = h_A(\beta/\varphi)^{g-2} \tag{35}$$

together with the fact that V is arbitrary lead to

$$a^\alpha{}_{;\alpha} = (\ln N)_{,\alpha} a^\alpha + 4(\ln N)_{,\alpha} n^\alpha + (6-3g)(\ln\varphi)_{,\alpha} n^\alpha$$
$$+ \int C(f)\pi + \int C(f)\ln(h^3 f/r)\pi \tag{36}$$

so

$$(s^\alpha/k)_{;\alpha} = n^\alpha{}_{;\alpha} - a^\alpha{}_{;\alpha} = (\ln N)_{,\alpha} s^\alpha/k$$
$$- [4\ln N + (6-3g)\ln\varphi]_{,\alpha} n^\alpha - \int C(f)\ln(h^3 f/r)\pi \tag{37}$$

Since the collision integral in (37) has precisely the same form as in standard theory, its contribution must be non-negative for precisely the same reasons as in standard theory [note that $\ln(h^3 f/r) < 0$]. Further since

$$\Pi(s^\alpha/k) = \Pi(n^\alpha) = -4 \tag{38}$$

$$(s^\alpha/k)_{;\alpha} = (s^\alpha/k)_{*\alpha} = (s^\alpha/k)_{\parallel\alpha}$$
$$\geqslant (\ln N)_{,\alpha} s^\alpha/k - [4\ln N + (6-3g)\ln\varphi]_{,\alpha} n^\alpha \tag{39}$$

which is the units covariant version of the H-theorem. This is the basis for the second law of thermodynamics. Notice that in standard physics the right-hand side of (39) is zero.

3. STATISTICAL PHYSICS

In statistical physics it is customary to deal with the phenomena of kinetic theory in a situation which closely approximates physical equilibrium. However, in this theory complete thermodynamic equilibrium can never exist. This is because in any sample of fluid there are always replicating particles. Also, the energy of every sample of fluid is always changing, regardless of how well insulated the fluid may be. The cosmic effect φ can never be turned off or screened out just as gravity can never be turned off or screened out.

However, when the relevant time scales are examined one can discuss sensibly an effective equilibrium. If the time scale for a small segment of fluid to reach thermal equilibrium is much smaller than the time scale t_0 for φ-generated effects to occur, then the fluid can be considered to pass slowly through a sequence of equilibrium states characterized by a total number of particles $N(t)$ and a temperature $T(t)$. Such an effective equilibrium is characterized by the vanishing of the collision integrals in (25), (29), and (37). Consequently, wherever in standard physics the Bose–Einstein, Fermi–Dirac, or Maxwell–Boltzmann distribution functions are valid, so too are they valid in this LNH formalism. The only difference is that now both the temperature and the chemical potential are time dependent (φ dependent). In the remainder of these papers the term "equilibrium" should be understood to mean equilibrium in this quasistatic sense.

Of course, just as in standard physics one must not set $C(f) = 0$ in the Boltzmann equation (18). This would lead one to conclude

$$f = f_0(\beta/\beta_0)^{6-3g}(N/N_0)^4 \tag{40}$$

which is valid only if the particles do not collide at all (Vlasov equation), cf. Paper II. Instead one asserts that the net result of collisions is to thermalize the new particles and energy generated through φ. This is equivalent to the case of detailed balancing in standard physics. The end result is that when local thermodynamic effective equilibrium holds one has

$$f = \frac{r/h^3}{e^{p^\lambda \gamma_\lambda - \alpha} \mp 1} \tag{41}$$

just as in standard physics. Here $\alpha(x)$ and $\gamma_\lambda(x)$ reduce to μ/kT and u_λ/kT, respectively, where u^λ is the mean fluid velocity.

In the statistical physics limit one has (Stewart, 1971)

$$n^\alpha = nu^\alpha \tag{42}$$

$$T^{\mu\alpha} = (\rho + p)u^\mu u^\alpha - pg^{\mu\alpha} + q^\mu u^\alpha + q^\alpha u^\mu \tag{43}$$

$$s^\alpha = snu^\alpha + q^\alpha/T - \mu\mathcal{J}^\alpha/T \tag{44}$$

where n is the number density of particles, u^α is the mean fluid velocity, ρ is the energy density, p the isotropic kinetic pressure, s the entropy per particle, T the kinetic temperature, μ the chemical potential, q^α the heat flux relative to u^α, and $\mathcal{J}^\alpha$ the diffusion flux of particles relative to u^α. Omitted from (43) are terms involving viscosity. For an observer traveling with the mean fluid velocity u^α equations (42), (43), (44), (20), (21), and (31) give

$$n = \int f\, d^3\mathbf{p} \tag{45}$$

$$\rho = \int Ef\, d^3\mathbf{p} \tag{46}$$

$$p = 1/3\int \frac{\mathbf{p}^2}{E} f\, d^3\mathbf{p} = 1/3\int \mathbf{p}\cdot\frac{\partial E}{\partial \mathbf{p}} f\, d^3\mathbf{p} \tag{47}$$

$$sn = kn - k\int f \ln(h^3 f/r)\, d^3\mathbf{p} \tag{48}$$

where $E \equiv u_\alpha p^\alpha$ is the energy per particle in the fluid rest frame.

In standard physics it is customary at this point to take the dilute fluid limit to obtain the Maxwell–Boltzmann distribution function. In this version of LNH the dilute fluid limit is mandatory for any system of fermions. This is simply because LNH theory at the level of quantum physics has not yet been created. The units covariant formalism developed in Paper I assumes that quanta are free to "replicate." This is possible only if they are not phase space limited, or in quantum language, only for nondegenerate fermion systems. Of course bosons are not restricted this way, which is why the treatment of photon replication in Paper II is expected to be valid.

With f given by

$$f = \frac{r}{h^3} e^{\mu/kt} e^{-E/kt} \tag{49}$$

in the dilute fluid limit, one finds

$$p = nkT = nm/\chi \tag{50}$$

$$\rho = nm[3K_3(\chi) + K_1(\chi)]/4K_2(\chi) \tag{51}$$

$$\chi \equiv m/kT \tag{52}$$

where $K_n(\chi)$ is the Bessel function of the second kind with limits

$$K_n(\chi) \to \left(\frac{\pi}{2\chi}\right)^{1/2} e^{-\chi}[1 + (4n^2 - 1)/8\chi + O(\chi^{-2})] \qquad \text{as } \chi \to \infty \tag{53a}$$

$$K_n(\chi) \to (n-1)!2^{n-1}\chi^{-n}[1 + O(\chi)] \qquad \text{as } \chi \to 0 \tag{53b}$$

Use of (25) and (29) in a comoving frame with vanishing collision integrals allows one to relate χ to N, β, and φ. This is done in Section 5.3 below.

4. THERMODYNAMICS

In this section I formulate the basic laws of units covariant thermodynamics including LNH. Start by discussing Boltzmann's constant k. This constant originates in the kinetic theory definition of entropy flux (30). Consider (30) without k. Then by (33) s^α/k has the same power as $n^\alpha = nu^\alpha$ so by (44) and (30) the "entropy" s/k is equal to the logarithm of the statistical weight of the system and has power zero. This is as expected for the "information content" of a system.

The reciprocal of the rate of change of the "entropy" of a system with respect to the energy of the system is the absolute thermodynamic temperature θ of the system. Since "entropy" has power zero, θ must have the same power as energy (mass)

$$\Pi(\theta) = 1 - g \tag{54}$$

In practice it is usual to measure temperature in thermometer units called degrees. If T is the temperature in thermometer units then $\theta = kT$. This is the origin of Boltzmann's constant.

Thermometer units have power zero. This is most easily seen by noting that thermometer units are always defined in terms of pure numbers, e.g.,

$$1\text{ K} \equiv (V_B - V_F)/100V_F \tag{55}$$

where V_B is the volume of a given amount of mercury at the same temperature as boiling water, and V_F is the volume of the same amount of mercury at the same temperature as freezing water. Hence

$$\Pi(T)=0 \tag{56}$$

$$\Pi(k)=1-g \tag{57}$$

k relates the average energy of a fluid particle to the thermometer temperature T. Since the fluid particles are atomic particles k is strictly an atomic constant. Hence

$$k=k_A(\beta/\varphi)^{g-1}, \qquad k_A=1.38\times 10^{-16}\,\text{ergs/K} \tag{58}$$

To remove k from thermodynamic relations it is customary to introduce k into the definition of entropy as in (30). This entropy (no quotes) now has the same power as k

$$\Pi(s)=1-g \tag{59}$$

Since thermodynamic relations are usually written using differentials or exterior derivatives d one needs to introduce the concept of units covariant exterior derivative. The usual exterior derivative is defined by

$$df \equiv f_{,\alpha}dx^{\alpha} \tag{60}$$

where f is a scalar field. In a sense this definition is an accident due to the fact that the partial derivative of a scalar field is coordinate covariant so (60) is coordinate covariant as written. Since coordinate covariance is the only dynamical universal gauge invariance in classical physics, (60) is an adequate definition.

However, this theory utilizes an additional universal gauge invariance, viz., units covariance. Hence all derivative operators must be both coordinate covariant and units covariant. The units covariant gradient operator was defined in Paper I. Here the units covariant exterior derivative is defined by

$$d_*f \equiv f_{*\alpha}dx^{\alpha} \tag{61}$$

which is units covariant since coordinates have power zero. Of course (61) is normalized so that in G units ($\beta=1$) $d_*=d$. It will be convenient to introduce a second units covariant derivative $d_{\parallel}$ normalized so that in A

units ($\beta = \varphi$) $d_{\parallel} = d$

$$d_{\parallel} f \equiv f_{\parallel \alpha} dx^{\alpha} \tag{62}$$

As mentioned at the end of Paper I this bar derivative is formed in exactly the same way as the star derivative except that β/φ replaces β everywhere. Thus

$$f_{\parallel \alpha} = f_{,\alpha} + \Pi(f) f [\ln(\beta/\varphi)]_{,\alpha} \tag{63}$$

if f is a scalar field. This means that if h is an arbitrary differentiable function then

$$[h(\beta/\varphi)]_{\parallel \alpha} = 0 \tag{64}$$

just as

$$[h(\beta)]_{*\alpha} = 0 \tag{65}$$

Now one can obtain the second law of thermodynamics in this units covariant formalism containing LNH. In a local comoving frame with no heat flux or particle diffusion flux (44) gives

$$s^{\alpha} = snu^{\alpha} \tag{66}$$

Substitution into (39) and use of (25), (57), and (59) give

$$(s/k)_{;\alpha} u^{\alpha} \geqslant -[4 \ln N + (6-3g) \ln \varphi]_{,\alpha} u^{\alpha} \tag{67}$$

or since s/k has power zero and $k_{\parallel \alpha} = 0$

$$s_{\parallel \alpha} u^{\alpha} \geqslant -k[4 \ln N + (6-3g) \ln \varphi]_{,\alpha} u^{\alpha} \tag{68}$$

In terms of exterior derivative notation (68) becomes

$$d_{\parallel} s \geqslant -k d_{\parallel} [4 \ln N + (6-3g) \ln \varphi] \tag{69}$$

Either of equations (68) or (69) constitutes the LNH units covariant form of the second law of thermodynamics.

In standard physics the first law of thermodynamics is written as

$$d\rho = \sum_{A} [(Ts_A + \mu_A) \, dn_A + Tn_A \, ds_A] \tag{70}$$

together with

$$\rho + p = \sum_A (Ts_A + \mu_A) n_A \tag{71}$$

where the sum is over all the particle species present. In the units covariant formulation (71) will be unchanged since it is already units covariant. However, one expects the units covariant generalization of (70) to consist of replacing d by d_* or $d_{\parallel}$. The correct choice is dictated by the observation that for atomic systems thermodynamics is based on kinetic theory which in turn is based on atoms. Hence (70) is valid in A units ($\beta = \varphi$) so

$$d_{\parallel}\rho = \sum_A [(Ts_A + \mu_A)\, d_{\parallel} n_A + Tn_A d_{\parallel} s_A] \tag{72}$$

together with

$$\rho_{\parallel\alpha} u^\alpha = \sum_A [(Ts_A + \mu_A) n_{A\parallel\alpha} + Tn_A s_{A\parallel\alpha}] u^\alpha \tag{73}$$

constitute the LNH units covariant form of the first law of thermodynamics. For a one-component gas (72) and (73) take the more familiar forms

$$d_{\parallel}\rho = \left(\frac{\rho + p}{n}\right) d_{\parallel} n + Tn\, d_{\parallel} s \tag{74}$$

$$\rho_{\parallel\alpha} u^\alpha = \left(\frac{\rho + p}{n}\right) n_{\parallel\alpha} u^\alpha + Tns_{\parallel\alpha} u^\alpha \tag{75}$$

after use of (71).

In terms of the macroscopic quantities

$$U \equiv \rho V, \qquad S \equiv Ns, \qquad N \equiv nV \tag{76}$$

where V is the system volume, equations (69), (71), and (72) become

$$d_{\parallel} S \geqslant S d_{\parallel} \ln N - 4Nk\, d_{\parallel} \ln N - 3(2-g) Nk\, d_{\parallel} \ln \varphi \tag{77}$$

$$U + pV = TS + \sum_A \mu_A N_A \tag{78}$$

$$d_{\parallel} U = T d_{\parallel} S - p\, d_{\parallel} V + \sum_A \mu_A d_{\parallel} N_A \tag{79}$$

respectively.

Since the second law is not of the form $d_{\parallel} S \geqslant 0$, a problem with terminology arises. Write

$$N \sim \varphi^{g-1-x(g+2)/4} \tag{80}$$

where $x = 0$ for matter and $x = 1$ for photons. Then (77) becomes

$$d_{\parallel} S \geqslant S\, d \ln N - Nk(1-x)(1+2/g)\, d \ln t \tag{81}$$

where the LNH approximation $\varphi^g \sim t$ has been used. For $g = +1$ the entropy of a system can actually decrease! For the canonical LNH with $g = -1$

$$d_{\parallel} S \geqslant S\, d \ln N + Nk(1-x)\, d \ln t > 0 \tag{82}$$

Since $d_{\parallel} S > 0$ there is no such thing as adiabatic change. Also, no transformation is reversible. I will refer to the minimum entropy change case as an ideal process. Then for any ideal process

$$d_{\parallel} s = -k\, d_{\parallel} [4 \ln N + (6-3g) \ln \varphi] \tag{83}$$

$$d_{\parallel} S = S\, d_{\parallel} \ln N - Nk\, d_{\parallel} [4 \ln N + (6-3g) \ln \varphi] \tag{84}$$

There is no physical means possible whereby $d_{\parallel} S$ can be reduced below (84) since that would involve screening the effects of φ, which is not possible.

5. APPLICATIONS

5.1. Specific Heats. As a trivial illustration of some of the differences between standard thermodynamics and this units covariant form with LNH I discuss the specific heats of an ideal gas. In the standard definition of specific heat one considers a thermodynamic system in equilibrium at temperature T and adds a small amount of heat $đQ$ to the system while keeping all other external parameters constant. Normally the rate at which heat is added to the system is not important. However, in this units covariant theory with LNH rates are very relevant to any given problem. The heat $đQ$ must be added to the system in such a manner that the system can equilibrate in a time very short compared with t_0, the time scale of variation of φ. Of course, since t_0 is roughly equal to the age of the Universe there is no problem for this example. However, since the effects of φ can never be shielded, no thermodynamic problem, not even ideal ones, can be phrased so that the system takes an arbitrarily long time to equilibrate.

When this limitation is taken into account then N, β, and φ are all effectively constant.

From the first law (79) one has

$$T dS = đQ = (g-1) TS\, d\ln(\beta/\varphi) + dU + (1-g) U d\ln(\beta/\varphi) + p\,dV + 3pV d\ln(\beta/\varphi) - \sum_A \mu_A N_A\, d\ln N \tag{85}$$

using (59) with

$$\Pi(N) = 0, \qquad \Pi(V) = 3 \tag{86}$$

where one assumes that one is not adding particles to the system so

$$dN_A = N_A\, d\ln N \tag{87}$$

Now

$$dU = \left(\frac{\partial U}{\partial T}\right)_{V,t} dT + \left(\frac{\partial U}{\partial V}\right)_{T,t} dV + U d\ln N + (g-1) U d\ln(\beta/\varphi) \tag{88}$$

where the $d\ln N$ term exists since U is proportional to the number of particles present, and the $d\ln \beta/\varphi$ term exists because U is an atomic quantity (normalized to $\beta = \varphi$) and has power $1-g$.

Equation (88) illustrates an important source of error in this kind of analysis. From (85) one obtains a $d\ln(\beta/\varphi)$ term from $d_{\parallel}U$. In (88) one finds the same term with opposite sign. While it may appear that the term is being counted twice, this is not so. The $d_{\parallel}$ exterior derivative treats β/φ as a constant. Hence it must appear in (88) with opposite sign to that in (85) so that it does not appear in the end result, as required by the $d_{\parallel}$ operator.

The analysis now proceeds as in standard physics. For a perfect gas

$$\left(\frac{\partial U}{\partial V}\right)_{T,t} = 0 \tag{89}$$

From (85) and (88)

$$\upsilon c_V \equiv \left(\frac{đQ}{dT}\right)_{V,t} = \left(\frac{\partial U}{\partial T}\right)_{V,t} \tag{90}$$

where υ is the number of moles and c_V is the specific heat per mole, so (88) becomes

$$dU = \upsilon c_V dT + U d\ln N + (g-1) U d\ln(\beta/\varphi) \tag{91}$$

From (85) and (91)

$$\upsilon c_p \equiv \left(\frac{dQ}{dT}\right)_{p,t} = \left(\frac{\partial U}{\partial T}\right)_{p,t} + \left(\frac{\partial V}{\partial T}\right)_{p,t} = \upsilon c_V + p\left(\frac{\partial V}{\partial T}\right)_{p,t} \tag{92}$$

and use of (50) gives

$$c_p = c_V + R \tag{93}$$

just as in standard physics (as expected). One immediately defines

$$\Gamma \equiv c_p/c_V \tag{94}$$

to be the ratio of specific heats.

5.2. Ideal Expansion or Compression. For an ideal change one has

$$TdS = (g-1)TS\,d\ln(\beta/\varphi) + TS\,d\ln N - pVd[4\ln N + (6-3g)\ln\varphi] \tag{95}$$

from (84) and (50). Equating (85) and (95) and using (91) gives

$$\upsilon c_V\,dT + p\,dV + 3pV d\ln(\beta N\varphi^{1-g}) = \left(TS + \sum_A \mu_A N_A - U - pV\right) d\ln N = 0 \tag{96a}$$

from (78). Dividing by pV and using (50) gives

$$d\ln T + (\Gamma-1)\,d\ln V + 3(\Gamma-1)\,d\ln(\beta N\varphi^{1-g}) = 0 \tag{96b}$$

with (94). For Γ constant (96b) integrates as

$$TV^{\Gamma-1}(\beta N\varphi^{1-g})^{3(\Gamma-1)} = \text{const} \tag{97}$$

For nonrelativistic perfect fluids $\Gamma = 5/3$ and $N \sim \varphi^{g-1}$ so

$$T_{\mathrm{NR}} = T_0(V_0/V)^{2/3}(\beta_0/\beta)^2 \tag{98}$$

while for relativistic perfect matter fluids $\Gamma = 4/3$ and $N \sim \varphi^{g-1}$ so

$$T_R = T_0(V_0/V)^{1/3}(\beta_0/\beta) \tag{99}$$

For a gas of matter particles enclosed in a box of fixed volume

$$T_{NR} = T_0(\beta_0/\beta)^2 \tag{100}$$

$$T_R = T_0(\beta_0/\beta) \tag{101}$$

This is completely consistent with the kinetic theory interpretation of temperature as being proportional to the mean kinetic energy of the gas. In Paper I it was shown that the velocity of a nonrelativistic particle satisfies $v\beta = \text{const}$, which leads to (100), while a relativistic particle has $\gamma\beta = \text{const}$, which leads to (101) ($\gamma^{-2} \equiv 1 - v^2$).

Finally, for a box of fixed volume with perfectly reflecting walls containing electromagnetic radiation (photons) one has from (97)

$$T_\gamma = T_{\gamma 0}(\beta_0\varphi/\beta\varphi_0)(\varphi/\varphi_0)^{(g-2)/4} \cong T_{\gamma 0}(\beta_0\varphi/\beta\varphi_0)(t/t_0)^{(1-2/g)/4} \tag{102}$$

where $N_\gamma \sim \varphi^{3(g-2)/4}$ was chosen so as to preserve a free-photon distribution function as shown in Paper II (cf. Section 5.4 below). Hence the radiation temperature in the box either increases or decreases as $1/g < 1/2$ or $1/g > 1/2$, respectively, in A units.

5.3. Matter Temperature in Cosmology. I calculate the cosmological matter temperature both from thermodynamics and from statistical physics. The Universe is taken as isotropic and homogeneous with scale factor $R(t)$. If one puts a small comoving box in the cosmic soup then no net heat flux crosses the box boundaries. Thus the matter inside the box is undergoing an ideal expansion so (97) is applicable. From (97)

$$T_{NR} = T_0(R_0\beta_0/R\beta)^2 \tag{103a}$$

$$T_R = T_0(R_0\beta_0/R\beta) \tag{103b}$$

where $V \sim R^3$.

One can also use the technique mentioned at the end of Section 3 based on statistical physics. In a comoving frame $T^{\mu\alpha}{}_{*\alpha} = 0$ [equation (29)] becomes

$$\frac{\dot{\rho}}{\rho}+3\frac{\dot{R}}{R}+(1-g)\frac{\dot{\beta}}{\beta}+3\frac{p}{\rho}\frac{(R\beta)^{\cdot}}{R\beta}=0 \tag{104}$$

from (43) with $q^\alpha = 0$ and the cosmological line element

$$d\tau^2 = dt^2 - R^2(t)h_{ij}dx^i dx^j \tag{105}$$

while (23) with (42) becomes

$$\frac{\dot{n}}{n}+3\frac{\dot{R}}{R}=\frac{\dot{N}}{N}=(g-1)\frac{\dot{\varphi}}{\varphi} \tag{106}$$

As usual the dot denotes time derivative. From (50) and (51)

$$\frac{P}{\rho}=\frac{1}{\chi}\left[\frac{4K_2(\chi)}{3K_3(\chi)+K_1(\chi)}\right] \tag{107}$$

while (51) gives

$$\frac{\dot{\rho}}{\rho}=\frac{\dot{n}}{n}+(g-1)\frac{\dot{\beta}}{\beta}+(1-g)\frac{\dot{\varphi}}{\varphi}+\frac{\partial}{\partial t}\ln\left[\frac{4K_2(\chi)}{3K_3(\chi)+K_1(\chi)}\right] \tag{108}$$

since n is for atomic masses. Use of (104), (106), and (107) in (108) gives

$$\frac{\partial}{\partial t}\ln\left[\frac{3K_3(\chi)+K_1(\chi)}{4K_2(\chi)}\right]+\frac{3}{\chi}\left[\frac{4K_2(\chi)}{3K_3(\chi)+K_1(\chi)}\right]\frac{\partial}{\partial t}\ln(R\beta)=0 \tag{109}$$

where

$$\frac{4K_2(\chi)}{3K_3(\chi)+K_1(\chi)}\to\begin{cases}\chi/3 & \text{as } \chi\to 0\\ 1-3/2\chi & \text{as } \chi\to\infty\end{cases} \tag{110}$$

Thus (108) becomes

$$(\ln\chi^{-1})^{\cdot}+[\ln(R\beta)]^{\cdot}=0 \qquad \text{as } \chi\to 0 \tag{111a}$$

$$\tfrac{3}{2}(\chi^{-1})^{\cdot}+3\chi^{-1}[\ln(R\beta)]^{\cdot}=0 \qquad \text{as } \chi\to\infty \tag{111b}$$

which have (103) as solutions. Note that (109) is exact for any temperature so the assumption of constant Γ used in deriving (97) is not necessary.

Finally I show that starting from the first and second laws of thermodynamics one can derive the relation

$$(\rho u^{\alpha})_{*\alpha} = -pu^{\alpha}{}_{*\alpha} \tag{112}$$

for the ideal expansion of a perfect fluid obtained in Paper I. This checks the self-consistency of the formulation. Using a comoving frame together with (68), (71), (73), (50), and (25) to get

$$\rho_{\parallel\alpha}u^{\alpha} = \left[\sum_{A}(Ts_A + \mu_A)n_A\right]\left[(\ln N)_{,\alpha}u^{\alpha} - u^{\alpha}{}_{\parallel\alpha}\right]$$

$$- p[4\ln N + (6-3g)\ln\varphi]_{,\alpha}u^{\alpha} \tag{113}$$

$$(\rho u^{\alpha})_{\parallel\alpha} = -pu^{\alpha}{}_{\parallel\alpha} + \rho(\ln N)_{,\alpha}u^{\alpha} - 3p[\ln N + (2-g)\ln\varphi]_{,\alpha}u^{\alpha} \tag{114}$$

which upon expanding the derivatives can be written as

$$(\rho u^{\alpha})_{*\alpha} = -pu^{\alpha}{}_{*\alpha} + (\rho - 3p)\left[\ln(N\varphi^{1-g})\right]_{,\alpha}u^{\alpha} = -pu^{\alpha}{}_{*\alpha} \tag{115}$$

since for matter $N \sim \varphi^{g-1}$ while for photons $\rho = 3p$.

5.4. Photon Temperature in Cosmology. If one can treat photons as a perfect fluid then the photon temperature for an ideal expansion follows (97) with $\Gamma = 4/3$. However, one cannot assume this *a priori*. The proper approach is to return to kinetic theory. As shown in Paper II the distribution function for a free expansion of photons from an initial black body distribution follows the law

$$f = A\left(\frac{\varphi}{\varphi_0}\right)^{4a+6-3g}\left(\frac{\beta\varphi_0}{\beta_0\varphi}\right)^{6-3g}(e^{h\nu/kT}-1)^{-1} \tag{116a}$$

$$\frac{h\nu}{kT} \equiv \frac{h_0\nu_0}{k_0T_0} \tag{116b}$$

where A and a are constants with $N_{\gamma} \sim \varphi^{a}$. Consequently, a black-body distribution (or any other distribution) is preserved in time in A units if and only if

$$a = \tfrac{3}{4}(g-2) \tag{117}$$

The observed 2.7 K cosmic black-body radiation and its interpretation as relict radiation of a free-photon expansion from very far in the past indicates that (117) is required.

If (117) holds then the black-body distribution is preserved and (116b) defines the way the radiation temperature changes with time. This follows since the argument of the exponential in (116a) determines the location of the peak of the Planck curve. This peak in turn *defines* the temperature of the black body through Wien's law. However, in Paper II I showed directly from the photon propagation equation that the frequency of a photon varies along its path like

$$vR\varphi^{a+2-g} = \text{const} \tag{118}$$

Combining (116b), (117), and (118) gives

$$T_\gamma = T_{\gamma 0}\left(\frac{R_0\beta_0}{R\beta}\right)\left(\frac{\varphi}{\varphi_0}\right)^{(2+g)/4} \tag{119}$$

in perfect agreement with (97) with $\Gamma = 4/3$.

6. DISCUSSION

This paper has presented a detailed discussion of the thermodynamics of atomic systems in the units covariant formalism including LNH. In order to properly develop thermodynamics one starts with kinetic theory, uses the in-geodesic equation of motion, and inserts particle creation rates where necessary. From the form of the collision integrals one concludes that equilibrium distributions have exactly the same form as in standard physics. This is vital. If at any point in this development φ affects the kinematics of a system then the dynamical significance of and difference between A units and G units is lost. φ can cause transitions among states but must not affect the states themselves.

Equilibrium considerations lead one to statistical physics. Here LNH is automatically incorporated in a self-consistent manner due to the development from kinetic theory. Taking the thermodynamic limit allows the units covariant formulation of the first and second laws of thermodynamics containing LNH. It is found that for canonical LNH no process can be either adiabatic or reversible. The term "ideal" is used for those thermodynamic processes for which dS is minimal. The net effect of φ on physical systems is as though all matter in our Universe is weakly coupled to a large energy source or sink (depending on the value of g).

A few simple applications of the theory were presented to illustrate its use and self-consistency. Cosmological matter and radiation temperatures were calculated and the kinetic theory results were shown to be in agreement with results based on the perfect fluid statistical physics approach.

It is to be emphasized that only the dilute fluid approximation should be valid. If matter is going to replicate then it must not be phase space limited. This means that none of this formulation is expected to be valid for degenerate fermion systems. The reason is simply that LNH has not yet been incorporated at the quantum level so there is no way to determine what happens when the Pauli principle dominates the dynamics. Hence this formulation of LNH is expected to be valid for internal processes in sufficiently high-temperature systems but is *not* expected to be valid for such "weird" cosmologically insignificant systems as planets, moons, or rocks.

Some systems for which LNH may well be significant are galaxy cluster dynamics, galaxy dynamics, and globular cluster dynamics. Although these are classical systems (stars do not replicate), the elements of each system (stars, galaxies) do follow in-geodesics and not geodesics. For canonical LNH this means that energy is being pumped into the system by φ. Over cosmological time scales (10^{10} yr) the amount of energy "produced" by φ can be comparable to the total initial energy. Of course, since G is decreasing and M increasing the effects are not easy to predict. However, notice that it is not correct to simply scale phenomena in terms of $G(t)$ and $M(t)$. One must also include the effects of the in-geodesic equation.

To a lesser extent this is also true of stellar evolution calculations. It is common to predict effects of LNH on stellar evolution by simple scaling with $G(t)$ and $M(t)$ (Canuto and Lodenquai, 1977; Canuto et al., 1977; Maeder, 1977; VandenBerg, 1977). To date the subtle effects of energy injection due to particles following in-geodesics have been ignored. Compared with nuclear energy sources the energy generation rate due to in-geodesic motion is miniscule. What is significant is that this energy is generated throughout the body of the star instead of being localized near the stellar core. It is precisely this effect which the formalism developed in this paper is designed to accommodate.

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